A modified tangent-gas approximation for two-dimensional steady flow

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SUMMARY

A set of two-dimensional subsonic flows past certain cylinders is obtained using hodograph methods, in which the true pressure-volume relationship is replaced by various straight-line approximations. It is found that the approximation obtained by a least-squares method possibly gives best results. Comparison is made with values obtained by using the von Kármán-Tsien approximation and also with results obtained by the variational approach of Lush & Cherry (1956).

1. INTRODUCTION

There are a number of ways available for finding the approximate steady two-dimensional flow past a fixed cylinder. One such method is the tangent-gas method proposed by Chaplygin and later extended by Tsien, which uses a tangent to the curve, pressure (p) *vs* volume (v), as a representation of the true relationship. The significance of this device is that the hodograph equations can be reduced to the Cauchy-Riemann differential equations. Demtchenko (1932) considered a tangent at the point on the curve corresponding to the stagnation conditions, and as a result his theory can only be applied, with any accuracy, to flows with speeds up to about one-half sonic speed. Tsien (1939), on von Kármán's suggestion, took the tangent at the point corresponding to conditions at infinity, assuming uniform flow there, and thus succeeded in extending the range of accuracy. These cases are only two examples of a set of flows obtainable by taking a tangent at each point of the (p, v)-curve, or, more precisely, a series of straight-line approximations to the curve within the significant limits for the particular problem.

A method is suggested for finding the particular line which possibly will yield the best results, and comparison is made with the accurate values obtained by Lush & Cherry (1956), using a variational method, for the adiabatic flow past a circular cylinder.

2. EQUATIONS OF MOTION

Assume

$$p = a - b^2 v \tag{1}$$

as the straight-line approximation to the (p, v)-curve, which for illustration may be taken to be the adiabatic relation

$$pv^{\gamma} = 1.$$

(For convenience the stagnation pressure and density are taken as unity.) Since $v = 1/\rho$ (ρ representing fluid density), Bernoulli's theorem immediately yields

$$q^2 = b^2(v^2 - 1), \tag{2}$$

where q represents fluid speed such that $v = 1$ when $q = 0$.

If ϕ represents the velocity potential and ψ the stream function, the hodograph equations are

$$\phi_q = q\frac{d}{dq}\left(\frac{1}{q\rho}\right)\psi_{\theta}, \qquad \phi_{\theta} = \left(\frac{q}{\rho}\right)\psi_q, \tag{3}$$

where θ is the angle the fluid velocity makes with a fixed direction.

Equations (3) can be reduced immediately to the Cauchy-Riemann differential equations

$$\phi_{\omega} = -\psi_{\theta}, \qquad \phi_{\theta} = \psi_{\omega}, \tag{4}$$

by using (2), and setting

$$\begin{aligned}\omega &= \int (\rho/q)\, dq \\ &= \log[Bq/\{b + (b^2 + q^2)^{1/2}\}],\end{aligned} \tag{5}$$

where B is an arbitrary constant.

From (4), $\phi + i\psi$ must be an analytic function of $\omega - i\theta$, that is to say,

$$W = \phi + i\psi = f(\bar{\sigma}),$$

with $\sigma = \omega + i\theta$ and the over-bar representing a complex conjugate.

If x, y represent coordinates in the plane of flow, then quite simply

$$dz = q^{-1}e^{i\theta}(d\phi + i\rho^{-1}\, d\psi), \qquad z = x + iy,$$

and, in particular, on a profile given by $\psi =$ constant,

$$dz = q^{-1}e^{i\theta}\, dW. \tag{6}$$

Now the function $f(\bar{\sigma})$ will represent the flow of an incompressible fluid in a certain ζ-plane, so that

$$\frac{dW}{d\zeta} = Qe^{-i\theta}, \qquad Q = Ce^{\omega},$$

where C is an arbitrary constant and Q, θ are respectively the magnitude and angle of inclination of the velocity vector of the incompressible fluid in the ζ-plane. Equation (5) shows that

$$Q = 2bq/\{b + (b^2 + q^2)^{1/2}\}, \tag{7}$$

if the condition $Q \to q$ as $q \to 0$ is assumed, so that

$$q = 4b^2Q/(4b^2 - Q^2). \tag{8}$$

Assume the flow in the ζ-plane to be uniform and parallel at infinity, and write $W = Q_{\infty}G(\zeta)$, Q_{∞} being the fluid speed at infinity; then

$$Q^2 = Q_{\infty}^2 \frac{dG}{d\zeta}\frac{d\bar{G}}{d\bar{\zeta}},$$

and from (6) and (8),

$$z = \zeta - \frac{Q_\infty^2}{4b^2} \int \left(\frac{d\bar{G}}{d\bar{\zeta}}\right)^2 d\bar{\zeta}, \tag{9}$$

since $dG = d\bar{G}$ along the profile ψ = constant.

3. Pressure and density

Let the straight line (1) be parallel to the tangent to the curve $pv^\gamma = 1$ at the point (p_1, v_1). Then

$$\left(\frac{dp}{dv}\right)_1 = \left(\frac{dp}{d\rho}\frac{d\rho}{dv}\right)_1 = -c_1^2\,\rho_1^2,$$

where c_1 is the local sonic speed. This linear assumption yields

$$c^2 = \frac{dp}{d\rho} = \frac{b^2}{\rho^2},$$

so that

$$b^2 = c^2\rho^2 = c_1^2\,\rho_1^2. \tag{10}$$

Equation (2) now gives

$$\rho^2 = 1 - M^2, \tag{11}$$

where $M = q/c$ is the Mach number. This means that the density and pressure are given very simply in terms of M, and as $\rho = 0$ when $M = 1$, it appears that (11) will probably not be a good approximation near the critical velocity.

It seems a reasonable assumption therefore to find only the fluid speed by this approximation method, and to use the correct adiabatic relations

$$p = \left(1 - \frac{\gamma-1}{2\gamma}q^2\right)^{\gamma/(\gamma-1)}, \qquad \rho = \left(1 - \frac{\gamma-1}{2\gamma}q^2\right)^{1/(\gamma-1)}, \tag{12}$$

for pressure and density once q has been determined.

4. Best approximation when $\gamma = 1{\cdot}4$

Obviously a set of flows can be constructed by taking values of b to correspond to tangents at various points on the (p, v)-curve between the stagnation and critical conditions. Such a set has been calculated by taking tangents at points on the curve corresponding to Mach numbers 0, 0·1, 0·2, ..., 1·0, and applied in each case to uniform flow past a circular cylinder of unit radius in the ζ-plane, so that

$$G(\zeta) = \zeta + 1/\zeta.$$

The Mach number at infinity in the z-plane is taken to be $M_\infty = 0{\cdot}35$, and since

$$q^2 = \gamma M^2/\{1 + \tfrac{1}{2}(\gamma - 1)M^2\},$$

$q_\infty = 0{\cdot}4091$, when γ has the value 1·4.

Table 1 shows the thickness ratio of the corresponding cylinder in the z-plane, also the ratio $q_{\max}/q_\infty$, $q_{\max}$ being the maximum speed on the profile, which occurs at the points where thickness is greatest. An inspection of this table shows that although thickness ratio increases with M,

q_{max}/q_∞ increases even more quickly, so that it is reasonable to suppose that for any given thickness ratio, the value of q_{max}/q_∞ is too small if the tangent corresponding to $M = 0$ is taken and too large if the tangent corresponding to $M = 1$ is used. The problem is now to select the 'best' tangent, or in effect the best value of b^2, since only the slope of the

M	b^2	Q_∞	Thickness ratio	q_{max}/q_∞
0	1·400	0·3975	1·041	2·191
0·1	1·383	0·3974	1·041	2·194
0·2	1·334	0·3970	1·042	2·201
0·3	1·257	0·3964	1·045	2·215
0·4	1·144	0·3953	1·050	2·239
0·5	1·044	0·3941	1·055	2·262
0·6	0·9224	0·3921	1·062	2·300
0·7	0·7989	0·3894	1·072	2·351
0·8	0·6797	0·3865	1·086	2·423
0·9	0·5798	0·3832	1·101	2·508
1·0	0·4687	0·3779	1·127	2·663

Table 1. Thickness ratio and q_{max}/q_∞ for various tangent approximations with $\gamma = 1{\cdot}4$.

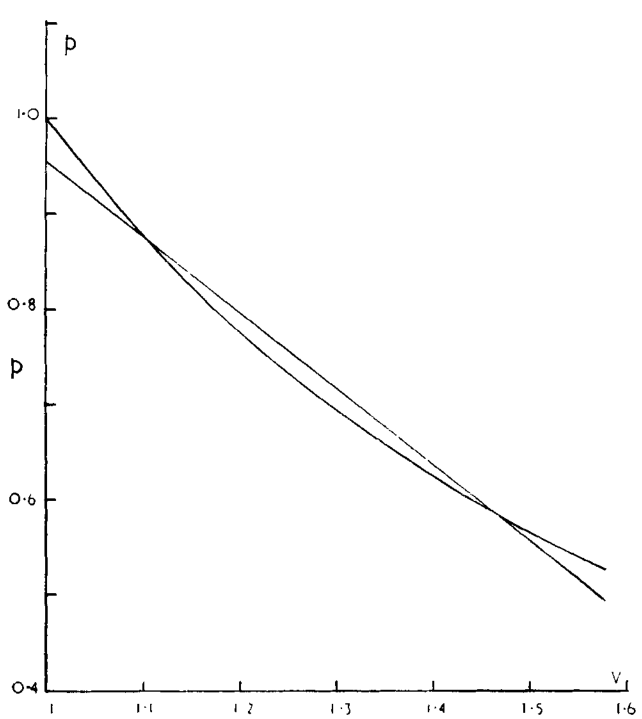

Figure 1. Approximation to (p, v)-curve.

straight-line approximation occurs in the results. Tsien (1939) chose the tangent at the point on the (p, v)-curve corresponding to the conditions at infinity, and it appears that the speeds obtained by this assumption are too small. It is suggested here that the value of b^2 should be taken to correspond to the best straight-line fit to the (p, v)-curve, found, say, by the method of least squares, between the significant limits for the particular problem under consideration. For example, it is found by using the least-squares method over 11 points between the stagnation and critical points, that the best straight-line fit over this range corresponds to $a = 1{\cdot}7544$, $b^2 = 0{\cdot}7982$. This is shown in figure 1.

5. Circular cylinder

In order to test the accuracy of this method, the compressible flow, uniform at infinity, past a circular cylinder of thickness ratio unity is considered, and comparison made with the accurate results obtained by Lush & Cherry (1956).

t	x	y	q/q_∞	$(x^2+y^2)^{1/2}$
0	4·325	0	0	4·325
10	4·247	0·811	0·3424	4·324
20	4·022	1·590	0·6846	4·325
30	3·661	2·298	1·0231	4·322
40	3·191	2·915	1·3516	4·322
50	2·643	3·423	1·6589	4·325
60	2·028	3·820	1·9271	4·325
70	1·370	4·103	2·1378	4·326
80	0·692	4·270	2·2716	4·326
90	0	4·325	2·3181	4·325

Table 2. Coordinates and fluid speed for a nearly circular cylinder at $M_\infty = 0{\cdot}4$.

To obtain such a flow in the z-plane, the uniform flow past an elliptic cylinder in the incompressible plane is considered, represented by

$$G(\zeta) = \zeta' + \alpha^2/\zeta',$$

where $\zeta = \zeta' + 1/\zeta'$; the profile is given by $\zeta' = \alpha e^{it}$. This is substituted in (9), from which it is found on integrating and separating into real and imaginary parts,

$$\left.\begin{aligned} x &= \left(\alpha + \frac{1}{\alpha}\right)\cos t - \lambda\left[\alpha(1+\alpha^2)\cos t + \tfrac{1}{4}(\alpha^2-1)^2 \log\left\{\frac{(\alpha^2-1)^2 + 4\alpha^2\sin^2 t}{(\alpha^2+2\alpha\cos t+1)^2}\right\}\right] \\ y &= \left(\alpha - \frac{1}{\alpha}\right)\sin t + \lambda\left[\alpha(1-\alpha^2)\sin t + \tfrac{1}{2}(\alpha^2-1)^2 \tan^{-1}\left\{\frac{2\alpha\sin t}{\alpha^2-1}\right\}\right] \end{aligned}\right\} \tag{13}$$

with $$\lambda = Q_\infty^2/4b^2.$$

The semi-axes of the resulting cylinder can be found by setting t equal to 0 and $\frac{1}{2}\pi$ in (13), and the ratio of these gives the thickness ratio δ as

$$\delta = \left(\frac{\alpha^2-1}{\alpha^2+1}\right)\left\{\frac{1+\lambda\left[-\alpha^2+\frac{1}{2}\alpha(\alpha^2-1)\tan^{-1}\left(\frac{2\alpha}{\alpha^2-1}\right)\right]}{1-\lambda\left[\alpha^2+\frac{1}{2}\alpha(\alpha^2-1)\left(\frac{\alpha^2-1}{\alpha^2+1}\right)\log\left(\frac{\alpha-1}{\alpha+1}\right)\right]}\right\}. \quad (14)$$

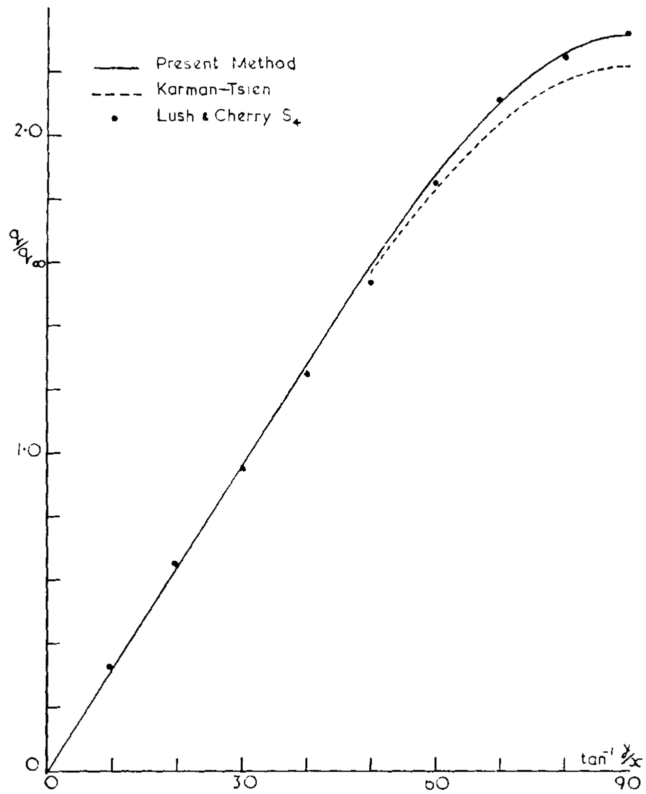

Figure 2. q/q_∞ plotted against $\tan^{-1}(y/x)$ for circular profile at $M_\infty = 0{\cdot}4$. Comparison is shown with the von Kármán-Tsien approximation and Lush & Cherry's S_4 result.

If now $M_\infty = 0{\cdot}4$, then $q_\infty = 0{\cdot}4659$, $Q_\infty = 0{\cdot}4380$, and, by employing the best straight-line fit previously found with $b^2 = 0{\cdot}7982$, λ takes the value 0·0601. Using this value of λ, it is found that when $\alpha^2 = 24{\cdot}00$, $\delta = 1$ to four significant figures. Table 2 shows coordinates, calculated from (13), and fluid speeds on the profile for various values of t. This table shows that the profile is almost circular, the error in radius being nowhere greater than 0·1%.

Figure 2 shows the graph of q/q_∞ plotted against $\tan^{-1}(y/x)$. From this an interesting comparison can be made with the results obtained by Lush & Cherry, and in particular with their S_4 approximation. Comparison is also made with a similar approach using the von Kármán-Tsien method. Agreement with Lush & Cherry is extremely good, although it must be observed that they use $\gamma = 1{\cdot}405$.

It would therefore appear that by this means of approximation a very accurate result can be obtained, at least for flow past cylinders with elliptic-type cross-section. Unfortunately there are very few solutions that are known to be sufficiently accurate for further comparison to be made.

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